

The number of ramified covering of a Riemann surface by Riemann surface

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Abstract

Interpreting the number of ramified covering of a Riemann surface by Riemann surfaces as the relative Gromov-Witten invariants and applying a gluing formula, we derive a recursive formula for the number of ramified covering of a Riemann surface by Riemann surface with elementary branch points and prescribed ramification type over a special point.

1 Introduction

Let Σ^g be a compact connected Riemann surface of genus g and Σ^h a compact connected Riemann surface of genus h ($g \geq h \geq 0$). A ramified covering of Σ^h of degree k by Σ^g is a non-constant holomorphic map $f : \Sigma^g \rightarrow \Sigma^h$ such that $|f^{-1}(q)| = k$ for all but a finite number of points $q \in \Sigma^h$, which are called branch points. Two ramified coverings f_1 and f_2 are said to be equivalent if there is a homeomorphism $\pi : \Sigma^g \rightarrow \Sigma^g$ such that $f_1 = f_2 \circ \pi$. A ramified covering f is called almost simple if $|f^{-1}(q)| = k - 1$ for each branch point but one, that is denoted by ∞ . If $\alpha_1, \dots, \alpha_m$ are the orders of the preimage of ∞ , then the ordered m -tuple pair $(\alpha_1, \dots, \alpha_m) = \alpha$ is a partition of k , denoted by $\alpha \vdash k$, and is called the ramification type of f (at ∞). We call m the length of α , denoted by $l(\alpha) = m$. Let $\mu_{h,m}^{g,k}(\alpha)$ be the number of equivalent almost simple covering of Σ^h by Σ^g with ramification type of α . How to determine $\mu_{h,m}^{g,k}(\alpha)$ is known as the Hurwitz Enumeration Problem. It is Hurwitz who first gave a explicit expression for $\mu_{0,m}^{0,k}(\alpha)$, see [H]. $\mu_{h,m}^{g,k}(\alpha)$ is called Hurwitz number. Many mathematicians contribute to this problem for the case $h = 0$. J.Dénes

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[D] gave a formula for $g = 0, l(\alpha) = 1$, and V.I.Arnol'd [A] for $g = 0, l(\alpha) = 2$. By a combinatorial method, I.P. Goulden and D.M. Jackson gave an explicit formula for $g = 0, 1, 2$. (see [GJ1,2,3], [GJV]). Physicists M.Crescimanno and W.Taylor [CT] solved the case when $g = 0$, and α is the identity. R.Vakil [V] derived explicit expressions for $g = 0, 1$, using a deformation theory of algebraic geometry .

In this paper we interpret Hurwitz number $\mu_{h,m}^{g,k}(\alpha)$ as the relative Gromov-Witten invariants defined by Li and Ruan [LR], then applying a gluing formula we derive a recursive formula for $\mu_{h,m}^{g,k}(\alpha)$ with $g \geq h$.

Suppose $\alpha = (\alpha_1, \dots, \alpha_m)$, put $\tilde{J}(\alpha) = \{(\alpha_i, \alpha_j) | 1 \leq i < j \leq m\} / \sim$, where $(\alpha_i, \alpha_j) \sim (\alpha_s, \alpha_t)$ iff $(\alpha_i, \alpha_j) = (\alpha_s, \alpha_t)$, or $(\alpha_i, \alpha_j) = (\alpha_t, \alpha_s)$. For every equivalent class $[(\alpha_i, \alpha_j)]$ of $\tilde{J}(\alpha)$ we choose a representative element, say (α_i, α_j) , and associate it a ordered $(m-1)$ -tuple $\theta = (\alpha_1, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_m, \alpha_i + \alpha_j)$, where the caret means that the term is omitted. Then we obtain a set $J(\alpha)$ of ordered $(m-1)$ -tuples. For $\theta = (\alpha_1, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_m, \alpha_i + \alpha_j) \in J(\alpha)$, we define a integer

$$(1.1) \quad I_1(\theta) = \begin{cases} \frac{1}{2}(\alpha_i + \alpha_j) \cdot \#\{\lambda \in \theta | \lambda = \alpha_i + \alpha_j\} & \text{if } \alpha_i = \alpha_j, \\ (\alpha_i + \alpha_j) \cdot \#\{\lambda \in \theta | \lambda = \alpha_i + \alpha_j\} & \text{if } \alpha_i \neq \alpha_j. \end{cases}$$

For every α_i in α , we construct a set $C_{\alpha_i}(\alpha) = \{\omega = (\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_m, \rho, \alpha_i - \rho) \mid 1 \leq \rho \leq [\frac{\alpha_i}{2}]\}$ of ordered $(m+1)$ -tuples. Let $\alpha_{1'}, \alpha_{2'}, \dots, \alpha_{l'}$ be all elements in α with distinct value. Put $C(\alpha) = C_{\alpha_{1'}}(\alpha) \cup C_{\alpha_{2'}}(\alpha) \cup \dots \cup C_{\alpha_{l'}}(\alpha)$. For every $\omega = (\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_m, \rho, \alpha_i - \rho) \in C(\alpha)$, we associate it a number

$$(1.2) \quad I_2(\omega) = \begin{cases} \frac{1}{2}\rho(\alpha_i - \rho) \cdot \#\{\lambda \in \omega | \lambda = \rho\} \cdot (\#\{\mu \in \omega | \mu = \alpha_i - \rho\} - 1) & \text{if } \rho = \alpha_i - \rho, \\ \rho(\alpha_i - \rho) \cdot \#\{\lambda \in \omega | \lambda = \rho\} \cdot \#\{\mu \in \omega | \mu = \alpha_i - \rho\} & \text{if } \rho \neq \alpha_i - \rho. \end{cases}$$

Dividing $\{1, 2, \dots, \hat{i}, \dots, m\}$ into two parts π_1, π_2 , correspondingly, we divide $\omega = (\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_m, \rho, \alpha_i - \rho)$ into two parts in forms: $\omega_{\pi_1} = (\alpha_{\pi_1}, \rho)$, $\omega_{\pi_2} = (\alpha_{\pi_2}, \alpha_i - \rho)$. For example if $\pi_1 = \{1\}$, $\pi_2 = \{2, \dots, \hat{i}, \dots, m\}$, then $\omega_{\pi_1} = (\alpha_1, \rho)$, $\omega_{\pi_2} = (\alpha_2, \dots, \hat{\alpha}_i, \dots, \alpha_m, \alpha_i - \rho)$. Note that π_1, π_2 may be empty. Write $\mathcal{P}_\omega = \{\pi = (\omega_{\pi_1}, \omega_{\pi_2})\} / \sim$, where $\pi = (\omega_{\pi_1}, \omega_{\pi_2}) = (\dots, \rho, \dots, \alpha_i - \rho) \sim \tilde{\pi} = (\tilde{\omega}_{\pi_1}, \tilde{\omega}_{\pi_2}) = (\dots, \tilde{\rho}, \dots, \alpha_i - \tilde{\rho}) \in \mathcal{P}_\omega$ iff $\rho = \tilde{\rho}$ and ω_{π_1} and $\tilde{\omega}_{\pi_1}$ are same through a permutation. We also use $\pi = (\omega_{\pi_1}, \omega_{\pi_2})$ to denote its equivalent class when there is no confusion. For every equivalent class $\pi = (\omega_{\pi_1}, \omega_{\pi_2}) \in \mathcal{P}_\omega$, we associate it a number

$$(1.3) \quad I_3(\pi) = \begin{cases} \frac{1}{2}\rho(\alpha_i - \rho) \cdot \#\{\lambda \in \omega_{\pi_1} | \lambda = \rho\} \cdot \#\{\mu \in \omega_{\pi_2} | \mu = \alpha_i - \rho\} & \text{if } \rho = \alpha_i - \rho, \\ \rho(\alpha_i - \rho) \cdot \#\{\lambda \in \omega_{\pi_1} | \lambda = \rho\} \cdot \#\{\mu \in \omega_{\pi_2} | \mu = \alpha_i - \rho\} & \text{if } \rho \neq \alpha_i - \rho. \end{cases}$$

In this paper, we will prove the following theorem, see Section 3:

Theorem A All $\mu_{h,m}^{g,k}(\alpha)$ can be determined by a recursive formula:

$$\begin{aligned}
(1.4) \quad \mu_{h,m}^{g,k}(\alpha) &= \sum_{\theta \in J(\alpha)} \mu_{h,m-1}^{g,k}(\theta) \cdot I_1(\theta) + \sum_{\omega \in C(\alpha)} \mu_{h,m+1}^{g-1,k}(\omega) \cdot I_2(\omega) \\
&+ \sum_{\omega \in C(\alpha)} \sum_{\substack{g_1 + g_2 = g \\ g_1, g_2 \geq 0}} \sum_{\substack{k_1 + k_2 = k \\ k_1, k_2 \geq 1}} \sum_{\substack{m_1 + m_2 = m+1 \\ m_1, m_2 \geq 1}} \sum_{\pi \in \mathcal{P}_\omega} \\
&\left(\begin{matrix} k+m-2kh-3+2g \\ k_1+m_1-2k_1h-2+2g_1 \end{matrix} \right) \mu_{h,m_1}^{g_1,k_1}(\omega_{\pi_1}) \cdot \mu_{h,m_2}^{g_2,k_2}(\omega_{\pi_2}) \cdot I_3(\pi).
\end{aligned}$$

where $m_i = l(\omega_{\pi_i}), i = 1, 2$.

Let $p = (p_1, p_2, p_3, \dots)$ be indeterminates, and $p_\alpha = p_{\alpha_1} \cdots p_{\alpha_m}$ for $\alpha = (\alpha_1, \dots, \alpha_m)$. Introduce a generating function for $\mu_{h,m}^{g,k}(\alpha)$

$$(1.5) \quad \Phi_h(u, x, z, p) = \sum_{\substack{k, m \geq 1 \\ g \geq 0}} \sum_{\substack{\alpha \vdash k \\ l(\alpha) = m}} \mu_{h,m}^{g,k}(\alpha) \cdot \frac{u^{k+m-2kh-2+2g}}{(k+m-2kh-2+2g)!} \cdot \frac{x^k}{k!} \cdot z^g p_\alpha.$$

Then after symmetrizing α_i and α_j in the variable $\theta = (\alpha_1, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_m, \alpha_i + \alpha_j) \in J(\alpha)$, and ρ and $\alpha_i - \rho$ in the variable $\omega, \gamma = (\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_m, \rho, \alpha_i - \rho)$ in (1.4), we have the following

Theorem B *The recursive formula (1.4) is equivalent to the the following partial differential equation :*

$$(1.6) \quad \frac{\partial \Phi_h}{\partial u} = \frac{1}{2} \sum_{i,j \geq 1} \left(ij p_{i+j} z \frac{\partial^2 \Phi_h}{\partial p_i \partial p_j} + ij p_{i+j} \frac{\partial \Phi_h}{\partial p_i} \frac{\partial \Phi_h}{\partial p_j} + (i+j) p_i p_j \frac{\partial \Phi_h}{\partial p_{i+j}} \right).$$

Remark By a combinatorial method, I. P. Goulden and D. M. Jackson have proven the above equation for the case $h = 0$. and gave explicit formula of $\mu_{h,m}^{g,k}(\alpha)$ for $h = 0$ and $g = 0, 1, 2$ (see ([GJ1, 2, 3], [GJV]):

$$(1.7) \quad \mu_{0,m}^{0,k}(\alpha_1, \dots, \alpha_m) = \frac{|c_\alpha|}{k!} (k+m-2)! k^{m-3} \prod_{i=1}^m \frac{\alpha_i^{\alpha_i}}{(\alpha_i - 1)!}$$

$$(1.8) \quad \mu_{0,m}^{1,k}(\alpha_1, \dots, \alpha_m) = \frac{|c_\alpha|}{24k!} (k+m)! \prod_{i=1}^m \frac{\alpha_i^{\alpha_i}}{(\alpha_i - 1)!} (k^m - k^{m-1} - \sum_{i=2}^m (i-2)! e_i k^{m-i}),$$

where c_α is the conjugacy class of the symmetric group \mathcal{S}_k on k symbols indexed by the partition type α of k , and e_i is the i -th elementary symmetric function in $\alpha_1, \dots, \alpha_m$.

For $h = 0$, using the recursive formula (1.4) with initial value

$$\mu_{0,1}^{g,1}(1) = \begin{cases} 1 & \text{if } g = 0 \\ 0 & \text{if } g \geq 1 \end{cases},$$

we calculate some values of $\mu_{0,m}^{g,k}(\alpha)$ with the aid of **Maple** for $g = 0, 1, \dots, 5$, $k = 3, 4, 5$, $m = 1, 2, \dots, 5$:

α	$\mu_{0,m}^{0,k}(\alpha)$	$\mu_{0,m}^{1,k}(\alpha)$	$\mu_{0,m}^{2,k}(\alpha)$	$\mu_{0,m}^{3,k}(\alpha)$	$\mu_{0,m}^{4,k}(\alpha)$	$\mu_{0,m}^{5,k}(\alpha)$
(3)	1	9	81	729	6561	59049
(1,2)	4	40	364	3280	29524	265720
(1,1,1)	4	40	364	3280	29524	265720
(4)	4	160	5824	209920	7558144	272097280
(1,3)	27	1215	45927	1673055	60407127	2176250895
(2,2)	12	480	17472	629760	22674432	816291840
(1,1,2)	120	5460	206640	7528620	271831560	9793126980
(1,1,1,1)	120	5460	206640	7528620	271831560	9793126980
(5)	25	3125	328125	33203125	3330078125	333251953125
(1,4)	256	35840	3956736	409108480	41394569216	4156871147520
(2,3)	216	26460	2748816	277118820	27762350616	2777408868780
(1,1,3)	1620	234360	26184060	2719617120	275661886500	27700994510280
(1,2,2)	1440	188160	20160000	2059960320	207505858560	20803767828480
(1,1,1,2)	8400	1189440	131670000	13626893280	1379375197200	138543794363520
(1,1,1,1,1)	8400	1189440	131670000	13626893280	1379375197200	138543794363520

which coincide to the formulas (1.7) and (1.8) for $g = 0, 1$, respectively.

To do similar calculation for $h > 0$, we have to calculate some special Hurwitz number $\mu_{h,m}^{g,k}(\alpha)$ for the case when $k + m - 2 + 2g - 2kh = 0$, which we will discuss in another paper.

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2 Relative GW-invariant

Let (M, ω) be a real $2n$ -dimension compact symplectic manifold with symplectic form ω , and Z^0, \dots, Z^p symplectic submanifolds of M with real codimension 2. Denote $Z = (Z^0, \dots, Z^p)$.

Let Σ^g be a compact connected Riemann surface of genus $g \geq 0$. Suppose $A \in H_2(M, \mathbf{Z})$, $k^i = \{k_1^i, \dots, k_{l_i}^i\}$ a set of positive integers, $i = 0, \dots, p$, denoted by $K = \{k^0, \dots, k^p\}$. Consider moduli space $\mathcal{M} = \mathcal{M}_{A,l}^{M,Z}(g, K)$ of pseudo-holomorphic maps $f : \Sigma^g \rightarrow M$ with marked points x_1, \dots, x_l ; $y_1^0, \dots, y_{l_0}^0; \dots; y_1^p, \dots, y_{l_p}^p \in \Sigma^g$ such that $[f(\Sigma^g)] = A$, and f is tangent to Z^i at $y_1^i, \dots, y_{l_i}^i$ with order $k_1^i, \dots, k_{l_i}^i$, $i = 0, \dots, p$. Denote $x = (x_1, \dots, x_l)$, $y^i = (y_1^i, \dots, y_{l_i}^i)$, $y = (y^0, \dots, y^p)$. Note that the intersection numbers $\#(A \cdot Z^i)$ are topological invariants, and $\sum_{j=1}^{l_i} k_j^i = \#(A \cdot Z^i)$. Moreover, since Z^i is a symplectic submanifold, if A can be expressed by the image of a nontrivial pseudo holomorphic map $f : \Sigma^g \rightarrow M$, the intersection number $\#(Z^i \cdot A) \geq 0$. Similarly to the Gromov-Uhlenbeck compactification for the pseudo-holomorphic maps, we compactify \mathcal{M} by $\overline{\mathcal{M}} = \overline{\mathcal{M}}_{A,l}^{M,Z}(g, K)$, the space of relative stable maps (for details see [LR]). We have two natural maps:

$$(2.1) \quad \begin{aligned} \Xi_{g,l} : \overline{\mathcal{M}} &\rightarrow M^l \\ (f, \Sigma^g, x, y, K) &\mapsto (f(x_1), \dots, f(x_l)) \end{aligned}$$

and

$$(2.2) \quad \begin{aligned} p : \overline{\mathcal{M}} &\rightarrow Z_0 \times \dots \times Z_p \\ (f, \Sigma^g, x, y, K) &\mapsto ((f(y_1^0), \dots, f(y_{l_0}^0)), \dots, (f(y_1^p), \dots, f(y_{l_p}^p))). \end{aligned}$$

Roughly, the relative GW-invariants are defined as

$$\psi_{A,g,l}^{M,Z}(\delta|\beta, K) = \int_{\overline{\mathcal{M}}} \Xi_{g,l}^* \Pi_i \delta_i \wedge p^* \Pi_j \beta_j$$

where $\delta = (\delta_1, \dots, \delta_l)$, $\delta_i \in H^*(M, \mathbf{R})$, $\beta = (\beta^0, \dots, \beta^p)$, $\beta^j = (\beta_1^j, \dots, \beta_{l_j}^j)$, $\beta_i^j \in H^*(Z^j, \mathbf{R})$ for any i . For precise definition, see [LR].

If Σ^g is not connected, suppose there exists c connected components $\Sigma^{g_1}, \dots, \Sigma^{g_c}$, then the genus g is defined to be its algebraic genus, i.e., $g = \sum_{i=1}^c g_i - c + 1$. Let \mathcal{P}_x be the set of all ordered partitions of $\{x_1, \dots, x_l\}$ into c parts. Each $\pi = (\pi_1, \dots, \pi_c) \in \mathcal{P}_x$ records which marked points $x = (x_1, \dots, x_l)$ go on each components $\Sigma^{g_1}, \dots, \Sigma^{g_c}$. Similarly, we can define $\sigma^i = (\sigma_1^i, \dots, \sigma_c^i) \in \mathcal{P}_{y^i}$. Corresponding to the partition of y , we define the partition of K , i.e., $\sigma^i = (\sigma_1^i, \dots, \sigma_c^i) \in \mathcal{P}_{K^i}$, and write $\sigma_i = (\sigma_i^0, \dots, \sigma_i^p)$, $\sigma = (\sigma_1, \dots, \sigma_c)$. π, σ induces a partition of δ, β , respectively. Denote the parameters over the component Σ^{g_i} by $x_{\pi_i}, y_{\sigma_i}, \delta_{\pi_i}, \beta_{\sigma_i}$. Suppose that $f_i : \Sigma \rightarrow M$ is a relative stable pseudo holomorphic map such that $[f_i(\Sigma^{g_i})] = A_i$, and $\sum_{i=1}^c A_i = A$. Then for given (A_1, \dots, A_c) , π and σ , the relative GW-invariant $\psi_{A,g,l}^{M,Z,c}(\delta|\beta; K)$ is defined by

$$\psi_{A,g,l}^{M,Z,c}(\delta|\beta; K)(\pi, \sigma) = \prod_{i=1}^c \psi_{A_i, g_i, l_i}^{M, Z_{\sigma_i}}(\delta_{\pi_i} | \beta_{\sigma_i}; K_{\sigma_i}).$$

Consider the linearization of $\bar{\partial}$ -operator

$$D_f = D_{\bar{\partial}_J(f)} : C^\infty(\Sigma, f^*TM) \rightarrow \Omega^{0,1}(f^*TM).$$

If we choose a proper weighted Sobolev norm over $C^\infty(\Sigma, f^*TM)$ and $\Omega^{0,1}(f^*TM)$, we have the following, see [LR]

Lemma 2.1 *D_f is a Fredholm operator with index*

$$(2.3) \quad \text{Ind } (D_f) = 2C_1(M)A + (2n - 6)(1 - g) + 2 \sum_{i=0}^p l_i - 2 \sum_{i=0}^p \sum_{j=1}^{l_i} k_j^i + 2l$$

and the relative GW-invariant $\psi_{g,l}^{M,Z}(\delta|\beta; K)$ is defined to be zero unless

$$(2.4) \quad \sum_{i=1}^l \deg \delta_i + \sum_{i=0}^p \sum_{j=1}^{l_i} \deg \beta_j^i = \text{Ind } (D_f).$$

Suppose that $H : M \rightarrow R$ is a proper periodic Hamiltonian function such that the Hamiltonian vector field X^H generates a circle action. By adding a constant, we can assume that zero is regular value. Then $H^{-1}(0)$ is a smooth submanifold preserved by circle action. The quotient $B = H^{-1}(0)/S^1$ is the famous symplectic reduction. Namely, B has an induced symplectic structure, so we can regard B as a symplectic submanifold of M with real codimension 2. We cut M along $H^{-1}(0)$. Suppose that we obtain two disjoint components M^\pm which have boundary $H^{-1}(0)$. We can collapse the S^1 -action on $H^{-1}(0)$ to obtain two closed symplectic manifolds \overline{M}^\pm . This procedure is called as symplectic cutting, see [L], [LR]. Without loss of generality, suppose \overline{M}^+ contains $Z^+ = (Z^0, \dots, Z^q)$ as submanifolds and \overline{M}^- contains $Z^- = (Z^{q+1}, \dots, Z^p)$, $q \leq p$. There is a map

$$\pi : M \rightarrow \overline{M}^+ \bigcup_B \overline{M}^-.$$

It induces a homomorphism

$$\pi^* : H^*(\overline{M}^+ \cup_B \overline{M}^-, R) \rightarrow H^*(M, R).$$

It was shown by Lerman [L] that the restriction of the symplectic structure ω on M^\pm such that $\omega^+|_B = \omega^-|_B$ is the induced symplectic form from symplectic reduction. By the Mayer-Vietoris sequence, a pair of cohomology classes $(\delta^+, \delta^-) \in H^*(\overline{M}^+, R) \otimes H^*(\overline{M}^-, R)$ with $\delta^+|_B = \delta^-|_B$ defines a cohomology class of $\overline{M}^+ \cup_B \overline{M}^-$, denoted by $\delta^+ \cup_B \delta^-$.

Consider the moduli space $\mathcal{M}^+ = \mathcal{M}_{A^+, l^+}^{+\overline{M}^+, Z^+, B, c^+}(g^+, K^+, \alpha^+)$ which consists of tuple $(\Sigma^{g^+}, x^+, y^+, e^+, K^+, \alpha^+, f^+)$ with properties:

- Σ^{g^+} has c^+ connected components;

- $f^+ : \Sigma^{g^+} \rightarrow \overline{M}^+$ is a pseudo holomorphic map;
- $[f^+(\Sigma^{g^+})] = A^+$;
- f^+ is tangent to $Z^+ = (Z_0, \dots, Z_q)$ at $y^+ = (y^0, \dots, y^q)$ with order $K^+ = (k^0, \dots, k^q)$;
- f^+ is tangent to B at $e^+ = (e_1^+, \dots, e_v^+)$ with order $\alpha^+ = (\alpha_1^+, \dots, \alpha_v^+)$.

Similarly, we can define $\mathcal{M}^- = \mathcal{M}_{A^-, l^-}^{-\overline{M}^-, Z^-, B, c^-}(g^-, K^-, \alpha^-)$ which consists of tuple $(\Sigma^{g^-}, x^-, y^-, e^-, K^-, \alpha^-, f^-)$. According to [LR], we can glue f^+ and f^- to obtain a pseudo holomorphic map $f : \Sigma^g \rightarrow M$. A little more precisely, we glue \overline{M}^+ and \overline{M}^- as above. If f^+ and f^- have same periodic orbits at each end, i.e, they have same orders as they tangent to symplectic submanifold B , we can glue the maps f^+ and f^- as $f^+ \# f^-$ after gluing the domain of Riemann surface Σ^{g^+} and Σ^{g^-} , which is the connected sum of Σ^{g^+} and Σ^{g^-} . Then perturbing map $f^+ \# f^-$, we can get an unique pseudo holomorphic map $f : \Sigma^g \rightarrow M$. In our paper, we always require that $\Sigma^{g^+} \# \Sigma^{g^-}$ is a connected Riemann surface. The following index addition formula is useful to our paper,

Lemma 2.2[LR]

$$\text{Ind}(D_{f^+}) + \text{Ind}(D_{f^-}) = (2n - 2)v + \text{Ind}D_f.$$

We also need a well known fact about genus of connected sum of Riemann surfaces:

Lemma 2.3 *The following equality is satisfied:*

$$(2.5) \quad g = g^+ + g^- + v - 1$$

where g is the genus of Σ^g , g^\pm is the algebraic genus of Σ^{g^\pm} , v is the number of end, i.e., the number of the points where we glue $\Sigma^{g^+}, \Sigma^{g^-}$.

According to theorem 5.8 of [LR], the relative GW-invariant $\psi_{A,l}^{M,Z}(\delta|\beta; K)$ can be expressed by the relative GW-invariants over each connected component. Precisely, using the notations of [LR], suppose that $\mathcal{C}_{g,l,K}^{J,A}$ is the set of indices:

- (1) The combinatorial type of $(\Sigma^\pm, f^\pm) : \{A_i^\pm, g_i^\pm, l_i^\pm, K_i^\pm, (\alpha_1^\pm, \dots, \alpha_v^\pm)\}, i = 1, \dots, v, \sum_{i=1}^v \alpha_i^\pm = \#(A \cdot B)$;
- (2) A map $\rho : \{e_1^+, \dots, e_v^+\} \rightarrow \{e_1^-, \dots, e_v^-\}$, where $(e_1^\pm, \dots, e_v^\pm)$ denote the puncture points of Σ^\pm , satisfying
 - (i) The map ρ is one-to-one;

(ii) If we identify e_i^+ and $\rho(e_i^+)$, then $\Sigma^+ \cup \Sigma^-$ forms a connected closed Riemann surface of genus g ;

(iii) $f^+(e_i^+) = f^-(\rho(e_i^+))$ and they have same order of tangency;

(iv) $((\Sigma^+, f^+), (\Sigma^-, f^-), \rho)$ represents the homology class A .

For given $C \in \mathcal{C}_{g,l,K}^{J,A}$ suppose that π_C^\pm, σ_C^\pm are partitions of $x^\pm, y^\pm, e^\pm, \delta^\pm, \beta^\pm, \alpha^\pm$ induced by C . Then we have the following: ([LR] Lemma 5.4 and Theorem 5.8)

Lemma 2.4 $\mathcal{C}_{g,l,K}^{J,A}$ is a finite set, and

$$(2.6) \quad \psi_{A,g}^{M,Z}(\delta|\beta; K) = \sum_{C \in \mathcal{C}_{g,l,K}^{J,A}} \psi_C(\delta|\beta; K),$$

where

$$(2.7) \quad \psi_C(\delta|\beta; K) = \|\alpha\| \sum \delta^{I,J} \psi_{A^+,g^+,l^+}^{\overline{M}^+,Z^+,B,c^+}(\delta^+|\beta^+; \rho_I; K^+, \alpha)(\pi_C^+, \sigma_C^+) \cdot \psi_{A^-,g^-,l^-}^{\overline{M}^-,Z^-,B,c^-}(\delta^-|\beta^-; \rho_J; K^-, \alpha)(\pi_C^-, \sigma_C^-),$$

where $\|\alpha\| = \alpha_1 \cdots \alpha_v$; $\delta^{IJ} = \delta^{I_1 J_1} \cdots \delta^{I_v J_v}$, $\delta^{I_i J_i}$ being the Kronecker symbol; and $\{\rho_1, \dots, \rho_s\}$ is an orthonormal basis of $H^*(B, \mathbf{R})$, $\rho_I = \{\rho_{I_1}, \dots, \rho_{I_v}\} \subset \{\rho_1, \dots, \rho_s\}$, $\rho_J = \{\rho_{J_1}, \dots, \rho_{J_v}\} \subset \{\rho_1, \dots, \rho_s\}$.

For convenience in application, we will rewrite Lemma 2.4 in following steps:

Step 1. We divide A into A^+ and A^- such $A = A^+ \cup_B A^-$.

Step 2. Suppose Σ^{g^\pm} have $a_i \geq 0$ end points with order $i \in \{1, \dots, \#(A \cdot B)\}$ such that $\sum_i i \cdot a_i = \#(A \cdot B)$, and $g = g^+ + g^- + \sum_i a_i - 1$. Denote $a = (a_1, a_2, \dots)$.

Step 3. Suppose that $\tau^\pm = (\pi^\pm, \sigma^\pm) \in \mathcal{P}_{x^\pm} \times \mathcal{P}_{y^\pm, e^\pm}$ record which marked points in $\{x^\pm, y^\pm, e^\pm\}$ go on each component $\Sigma^{g_1^\pm}, \dots, \Sigma^{g_{c^\pm}^\pm}$, satisfying:

(1). $g^\pm = \sum_{i=1}^{c^\pm} g_i^\pm - c^\pm + 1, g_i^\pm \geq 0, i = 1, \dots, c^\pm$

(2). $f_i: \Sigma^{g_i^\pm} \longrightarrow \overline{M}^\pm$ are relative stable holomorphic maps, and $[f_i^+(\Sigma^{g_i^\pm})] = A_i^\pm, i = 1, \dots, c^\pm$

with $\sum_{i=1}^{c^\pm} A_i^\pm = A^\pm$.

Denote $\tau = (\tau^+, \tau^-)$. Note that τ^\pm induce a partition of δ^\pm, β^\pm, a and Z^\pm .

Step 4. For given a and τ , we glue Σ^{g^+} and Σ^{g^-} in above manner such that $\Sigma^{g^+} \# \Sigma^{g^-}$ is a connected Riemann surface of genus g . However, for given such a and τ , we can glue Σ^{g^+} and Σ^{g^-}

in many different ways such that $\Sigma^{g^+} \# \Sigma^{g^-}$ is a connected Riemann surface of genus g . Denote the number of different ways by $\kappa(a, \tau)$.

Then we have the following gluing formula for the relative GW-invariants :

Lemma 2.4'

$$(2.8) \quad \psi_{A,g}^{M,Z}(\delta|\beta; K) = \sum \|a\| \cdot \delta^{IJ} \cdot \sum_{\tau} \kappa(a, \tau) \cdot \psi_{A^+,g^+,l^+}^{\overline{M}^+,Z^+,B,c^+}(\delta^+|\beta^+; \rho_I; K^+, a)(\pi^+, \sigma^+) \cdot \psi_{A^-,g^-,l^-}^{\overline{M}^-,Z^-,B,c^-}(\delta^-|\beta^-; \rho_J; K^-, a)(\pi^-, \sigma^-),$$

where $\|a\| = 1^{a_1} \cdot 2^{a_2} \cdots$, and the first \sum denotes that we sum all possibility for

$$(2.9) \quad \begin{aligned} A &= A^+ \cup_B A^-, \\ g &= g^+ + g^- + v - 1, \\ \#(A \cdot B) &= \sum_i i \cdot a_i, \\ \rho_I, \rho_J &\subset \{\rho_1, \cdots, \rho_s\}. \end{aligned}$$

3 Relative GW-invariant over Σ^h

In our case, M is the Riemann surface Σ^h with real two dimension, thus Z consists of points, which are the divisors of M . Since $H_2(\Sigma^h, \mathbf{Z}) \cong \mathbf{Z}$, denoted the generator by H , then the first Chern class $C_1(\Sigma^h) = (2 - 2h)H$. Let $A = kH$. When we say $f \in \mathcal{M}_{A,l}^{\Sigma^h,Z}(g, K)$, we mean that $f : \Sigma^g \rightarrow \Sigma^h$ is a pseudo holomorphic map such that $[f(\Sigma^g)] = kH$ and there exists marked points $x, y \in \Sigma^g$, f is tangent to Z at y with order K . Note that $\sum_{j=1}^{l_i} k_j^i = \#(A \cdot Z_i) = \deg(f) = k$, then k^i is a partition of $k, i = 0, \cdots, p$. Moreover the relative GW-invariant $\psi_{A,g,0}^{\Sigma^h,Z}(|\beta; K) = 0$ unless

$$(3.1) \quad \sum_{i=0}^p \sum_{j=1}^{l_i} \deg \beta_j^i = 2C_1(\Sigma^h)A + 4(g-1) + 2 \sum_{i=0}^p l_i - 2 \sum_{i=0}^p \sum_{j=1}^{l_i} k_j^i$$

where $\beta_j^i \in H^*(Z^i, R)$, $j = 1, \cdots, l_i$, $i = 0, \cdots, p$. However, since Z^i is a point, $\deg \beta_j^i = 0$. Thus we have

$$(3.2) \quad (2 - 2h)k + 2(g-1) + \sum_{i=0}^p l_i - \sum_{i=0}^p \sum_{j=1}^{l_i} k_j^i = 0.$$

Remark 3.1 The equality (3.2) is exactly the Riemann-Hurwitz formula.

Suppose Σ^g is connected, choose $l = 0, K = (2, 1, \dots, 1; \dots; 2, 1, \dots, 1; \alpha_1, \dots, \alpha_m)$. Then by definition $\mu_{h,m}^{g,k}(\alpha)$ is just the relative GW-invariant $\psi_{A,g,0}^{\Sigma^h, Z}(|\beta; K)$.

From (3.1), we derive $p = k + m - 2kh - 2 + 2g$, i.e., we have $k + m - 2kh - 2 + 2g$ double branch points over Σ^h , otherwise, $\mu_{h,m}^{g,k}(\alpha) = 0$.

Now, we can prove theorem A by symplectic cutting and the gluing formula (2.8). We perform the symplectic cutting over Σ^h at ∞ in a small neighborhood such that there is only one other double branch point G in this neighborhood. We have

$$\overline{M}^+ = S^2, \quad \overline{M}^- = \Sigma^h$$

It's easy to observe that $A^+ = kH', A^- = kH$, where H' is the generator of $H_2(S^2, \mathbf{Z}) \cong \mathbf{Z}$. We may consider dimension condition equations:

$$(3.3) \quad \begin{cases} (k - m) + (2 - 1) + (k - v) = 2k - 2 + 2g^+ \\ g = g^+ + g^- + v - 1 \end{cases}$$

We first consider $\overline{M}^+ = S^2$. The map $f^+ : \Sigma^{g^+} \rightarrow \overline{M}^+$ branches at only three points: infinity, the fixed double branch point G and the symplectic reduction point B . Suppose $\Sigma^{g^+} = \cup_{i=1}^{c^+} \Sigma^{g_i^+}$, i.e., Σ^{g^+} has c^+ -connected components. Suppose the holomorphic map $u_i^+ : \Sigma^{g_i^+} \rightarrow \overline{M}^+$ has degree k_i^+ . It is obvious that $k_i^+ \leq k$. If $\Sigma^{g_i^+}$ contains a double ramification point, we have from Riemann-Hurwitz formula over this component that

$$(3.4) \quad k_i^+ - v_i^+ + 2 - 1 + k_i^+ - m_i^+ = 2k_i^+ - 2 + 2g_i^+$$

where $v_i^+ \geq 1, m_i^+ \geq 1$ is the number of ramification at the symplectic reduction point B and infinity, respectively. Note that the geometric genus $0 \leq g_i^+ \leq g$, so we have two cases: $(v_i^+, m_i^+, g_i^+) = (1, 2, 0)$, or $(2, 1, 0)$. By the same reason, if the component $\Sigma^{g_i^+}$ doesn't contain any double ramification point, we have one case $(v_i^+, m_i^+, g_i^+) = (1, 1, 0)$. Note that $\sum_{i=1}^{c^+} m_i^+ = m$, we have $v = \sum_{i=1}^{c^+} v_i^+ = m - 1$, or $m + 1$, correspondingly, $c^+ = m - 1$, or m . In sum, we have proven

Lemma 3.2 *For $\overline{M}^+ = S^2$, the holomorphic map $f_i : \Sigma^{g_i^+} \rightarrow \overline{M}^+$ has one of the following branch types*

- (1) $(\alpha_i; 1, 1, \dots, 1; \alpha_i), 1 \leq i \leq m$
- (2) $(\alpha_k, \alpha_l; 2, 1, \dots, 1; \alpha_k + \alpha_l), 1 \leq k < l \leq m$
- (3) $(\alpha_i; 2, 1, \dots, 1; \rho, \alpha_i - \rho), 1 \leq i \leq m, 1 \leq \rho \leq [\frac{\alpha_i}{2}]$

at infinity, the fixed branch point G and the symplectic reduction point B respectively.

If $v = m - 1$, then $c^+ = m - 1, g^+ = \sum_{i=1}^{c^+} g_i^+ - c^+ + 1 = 2 - m$. Substituting into $g = g^+ + g^- + v - 1$, we have $g^- = g$. Note that $g^- = \sum_{i=1}^{c^-} g_i^- - c^- + 1, 0 \leq \sum g_i^- \leq g, g_i^- \geq 0, c^- \geq 1$, we have only one case: $c^- = 1, g^- = g$. If $v = m + 1$, then $c^+ = m$. By the

same reason, we have two cases: $c^- = 1, g^- = g - 1$ and $c^- = 2, g^- = g_1^- + g_2^- - 2 + 1 = g - 1$, $g_1^+ \geq 0, g_2^- \geq 0$. In sum, we have

Lemma 3.3 *The genus g^- and the number c^- of connected components of Σ^- are one of the following cases:*

- (i) $c^- = 1, g^- = g$;
- (ii) $c^- = 1, g^- = g - 1$;
- (iii) $c^- = 2, g^- = g_1^- + g_2^- - 1 = g - 1, g_1^- \geq 0, g_2^- \geq 0$.

Regarding the symplectic reduction point $B \in \overline{M}^-$ as infinity, we get many new almost simple ramified covering maps $f_i^- : \Sigma^{g_i^-} \rightarrow \overline{M}^-$. However in any above case, the holomorphic map $f_i^- : \Sigma^{g_i^-} \rightarrow \overline{M}^-$ has either strict smaller number of ramification points at infinity, or strict smaller degree, or strict smaller genus than the holomorphic map $f : \Sigma^g \rightarrow M = \Sigma^h$. Thus if we have known the relative GW-invariant in \overline{M}^+ , by Lemma 2.4, we can get a recursive formula for $\mu_{h,m}^{g,k}(\alpha)$. We also need the following lemma.

Lemma 3.4 *Let $\theta = (\alpha_1, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_m, \alpha_i + \alpha_j) \in J(\alpha)$, $\omega = (\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_m, \rho, \alpha_i - \rho) \in C(\alpha)$ as in the introduction, then for \overline{M}^+ , the product $\psi_J(\alpha, \theta)$ of the relative GW-invariants of $(m - 1)$ connected components is*

$$(3.5) \quad \psi_J(\alpha, \theta) = \begin{cases} \frac{1}{\alpha_1} \dots \frac{\hat{1}}{\alpha_i} \dots \frac{\hat{1}}{\alpha_j} \dots \frac{1}{\alpha_m} & \text{if } \alpha_i \neq \alpha_j \\ \frac{1}{\alpha_1} \dots \frac{\hat{1}}{\alpha_i} \dots \frac{\hat{1}}{\alpha_j} \dots \frac{1}{\alpha_m} \cdot \frac{1}{2} & \text{if } \alpha_i = \alpha_j \end{cases}$$

and the product $\psi_C(\alpha, \omega)$ of the relative GW-invariants of m connected components is

$$(3.6) \quad \psi_C(\alpha, \omega) = \begin{cases} \frac{1}{\alpha_1} \dots \frac{\hat{1}}{\alpha_i} \dots \frac{1}{\alpha_m} & \text{if } \rho \neq \alpha_i - \rho, \\ \frac{1}{\alpha_1} \dots \frac{\hat{1}}{\alpha_i} \dots \frac{1}{\alpha_m} \cdot \frac{1}{2} & \text{if } \rho = \alpha_i - \rho. \end{cases}$$

Proof By the definition of the relative GW-invariant and Lemma 3.2, we only need to calculate the connected relative GW-invariant of two type:

$$(3.7) \quad \begin{aligned} Q_1 &= \psi_{kH,0,0}^{S^2,pt,pt,pt}(|pt, pt, pt; k; 1, 1, \dots, 1; k), k \geq 1 \\ Q_2 &= \psi_{kH,0,0}^{S^2,pt,pt,pt}(|pt, pt, pt; k; 2, 1, \dots, 1; \rho, k - \rho), \rho \geq 1, \end{aligned}$$

where the “pt” in the bracket records the point homology Poincare dual to the generator $E \in H^0(pt, R)$, and the others correspond to Z .

Regarding a holomorphic map $f : S^2 \rightarrow S^2$ as a meromorphic function over Riemann plane, we write $f \in \overline{\mathcal{M}}_{kH,0}^{S^2,pt,pt,pt}(0; K; 1, 1, \dots, 1; k)$ in the form $F_1 : C \rightarrow C$, $F_1(x) = \frac{\alpha_0(x-y^1)^k}{(x-y^2)^k}, x \in C$,

where $y_1 \neq y_2 \in C$ are k -ramification points. Without loss of generality, we choose zero and infinity as k -ramification points, and send 1 to 1, thus there exists a unique solution $F_1(x) = x^k$. However we have conformal transformation $\pi_i : C \rightarrow C, \pi_i(x) = e^{\frac{2\pi i}{k}} x, i = 0, \dots, k-1$, such that $F_1(x) = F_1 \circ \pi_i(x)$, i.e, there exists a finite group \mathbf{Z}_k that acts on $\overline{\mathcal{M}}_{kH,0}^{S^2,pt,pt,pt}(0; K; 1, 1, \dots, 1; k)$, thus

$$(3.8) \quad Q_1 = \frac{1}{k}.$$

By the same reason, we write $f \in \overline{\mathcal{M}}_{kH,0}^{S^2,pt,pt,pt}(0; K; 2, 1, \dots, 1; \rho, k - \rho)$ in form $F_2 : C \rightarrow C$,

$$F_2(x) = \frac{\alpha_0(x - y_1^1)^\rho(x - y_2^1)^{k-\rho}}{(x - y^2)^k},$$

$\alpha_0 \neq 0, x \in C$, where $y_1^1 \neq y_2^1, y^2 \in C$ are $\rho, k - \rho, k$ -ramification point, respectively. Suppose 1, 2 and zero are $\rho, k - \rho, k$ -ramification point, respectively. Then $F_2 = \frac{\alpha_0(x-1)^\rho(x-2)^{k-\rho}}{x^k}$. Since F_2 has a double ramification point x at a given point, for instance at 1. Then $x \neq 0, 1, 2$. We have following equation

$$(3.9) \quad \begin{cases} F_2(x) &= 1, \\ F_2'(x) &= 0. \end{cases}$$

Solving(3.8), we have unique solution $F_2 = \frac{\alpha_0(x-1)^\rho(x-2)^{k-\rho}}{x^k}$, where $\alpha_0 = \frac{(2k)^k}{\rho^\rho(2\rho-2k)^{k-\rho}}$. However, if $\rho = k - \rho$, we have conformal transformation $\pi : C \rightarrow C, \pi(x) = \frac{2x}{3x-2}$, such that $F_2(x) = F_2 \circ \pi(x)$. Since $\pi \circ \pi = \mathbf{1}$, there exists a finite group \mathbf{Z}_2 that acts on $\overline{\mathcal{M}}_{kH,0}^{S^2,pt,pt,pt}(0; K; 2, 1, \dots, 1; \rho, k - \rho)$, thus

$$(3.10) \quad Q_2 = \begin{cases} \frac{1}{2} & \text{if } \rho = k - \rho, \\ 1 & \text{if } \rho \neq k - \rho. \end{cases}$$

We complete the Lemma 3.4. \square

Now, we prove theorem A:

Theorem A *Hurwitz number $\mu_{h,m}^{g,k}(\alpha)$ can be determined by a recursive formula:*

$$(3.11) \quad \begin{aligned} \mu_{h,m}^{g,k}(\alpha) &= \sum_{\theta \in J(\alpha)} \mu_{h,m-1}^{g,k}(\theta) \cdot I_1(\theta) + \sum_{\omega \in C(\alpha)} \mu_{h,m+1}^{g-1,k}(\omega) \cdot I_2(\omega) \\ &+ \sum_{\omega \in C(\alpha)} \sum_{\substack{g_1 + g_2 = g \\ g_1, g_2 \geq 0}} \sum_{\substack{k_1 + k_2 = k \\ k_1, k_2 \geq 1}} \sum_{\substack{m_1 + m_2 = m + 1 \\ m_1, m_2 \geq 1}} \sum_{\pi \in \mathcal{P}_\omega} \\ &\left(\begin{matrix} k + m - 2kh - 3 + 2g \\ k_1 + m_1 - 2k_1h - 2 + 2g_1 \end{matrix} \right) \mu_{h,m_1}^{g_1,k_1}(\omega_{\pi_1}) \cdot \mu_{h,m_2}^{g_2,k_2}(\omega_{\pi_2}) \cdot I_3(\pi). \end{aligned}$$

where $m_i = l(\omega_{\pi_i}), i = 1, 2$.

Proof For a positive integer b and an ordered positive integer tuple $\beta = (\lambda_1, \dots, \lambda_t)$, we define an integer $\varphi(\beta, b) = \#\{\lambda \in \beta | \lambda = b\}$. According to Lemma 2.4' and Lemma 3.3, we have

$$\begin{aligned}
& \mu_{h,m}^{g,k}(\alpha) \\
&= \sum_{\theta \in J(\alpha)} \mu_{h,m-1}^{g,k}(\theta) \cdot \psi_J(\alpha, \theta) \cdot \|\theta\| \cdot \varphi(\theta, \alpha_i + \alpha_j) \\
&+ \sum_{\omega \in C(\alpha)} \mu_{h,m+1}^{g-1,k}(\omega) \cdot \psi_C(\alpha, \omega) \cdot \|\omega\| \cdot \varphi(\omega, \rho) \cdot (\varphi(\omega, \alpha_i - \rho) - \delta^{\rho, \alpha_i - \rho}) \\
(3.12) \quad &+ \sum_{\omega \in C(\alpha)} \sum_{\substack{g_1 + g_2 = g \\ g_1, g_2 \geq 0}} \sum_{\substack{k_1 + k_2 = k \\ k_1, k_2 \geq 1}} \sum_{\substack{m_1 + m_2 = m + 1 \\ m_1, m_2 \geq 1}} \sum_{\pi \in \mathcal{P}_\omega} \binom{k + m - 2kh - 3 + 2g}{k_1 + m_1 - 2k_1h - 2 + 2g_1} \\
&\cdot \mu_{h,m_1}^{g_1,k_1}(\omega_{\pi_1}) \cdot \varphi(\omega_{\pi_1}, \rho) \cdot \mu_{h,m_2}^{g_2,k_2}(\omega_{\pi_2}) \cdot \varphi(\omega_{\pi_2}, \alpha_i - \rho) \cdot \psi_C(\alpha, \omega) \cdot \|\omega\|
\end{aligned}$$

where $\|\theta\| = \alpha_1 \cdots \hat{\alpha}_i \cdots \hat{\alpha}_j \cdots \alpha_m(\alpha_i + \alpha_j)$; $\|\omega\| = \alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_m \rho(\alpha_i - \rho)$; $\delta^{\rho, \alpha_i - \rho}$ is the Kronecker symbol; $m_i = l(\omega_{\pi_i})$, $i = 1, 2$; the factor $\binom{k+m-2kh-3+2g}{k_1+m_1-2k_1h-2+2g_1}$ comes from the fact that we can choose $k_1 + m_1 - 2k_1h - 2 + 2g_1$ double ramification points over the component Σ^{g_1} from $k + m - 2kh - 3 + 2g$ double ramification points. Substituting (3.4), (3.5) into (3.11), we get (3.10). \square

Remark For $h = 0$, the initial value of our recursive formula is

$$\mu_{0,1}^{g,1}(1) = \begin{cases} 1 & \text{if } g = 0 \\ 0 & \text{if } g \geq 1 \end{cases},$$

For $h > 0$, the initial value is not only

$$\mu_{h,1}^{g,1}(1) = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{if } g \geq h + 1 \end{cases},$$

but also some special Hurwitz number $\mu_{h,m}^{g,k}(\alpha)$ for the case when $k + m - 2kh - 2 + 2g = 0$, which we will discuss in another paper.

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